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Spinning gas clouds with precession: a new formulation

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Abstract
We consider Dyson's model (Dyson F J 1968 *J. Math. Mech.* **18** 91) of an ellipsoidally stratified ideal gas cloud expanding adiabatically into a vacuum, in the Liouville integrable case where the gas is monatomic ($\gamma = 5/3$) and there is no vorticity (Gaffet B 2001a *J. Phys. A: Math. Gen.* **34** 2097; Paper I). In the cases of rotation about a fixed axis the separation of variables can be achieved, and the separable variables are linearly related to a set of three variables denoted by ρ, R, W (Gaffet B 2001b *J. Phys. A: Math. Gen.* **34** 9195; Paper II). We show in the present work that these variables admit a natural generalization to cases of rotation about a movable axis (precessing motion). The present study is restricted to the consideration of the so-called degenerate cases (see Gaffet B 2006 *J. Phys. A: Math. Gen.* **39** 99; Paper III), but we hope to generalize our results in the future to the non-degenerate ones as well. We also present a new, compact and generally valid formulation of one of the integrals of motion, of the sixth degree in the momenta, denoted by I_6.

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1. Introduction

The model of a gas cloud considered here belongs to the more general class of self-gravitating liquid and gas ellipsoid models whose study originated with Dirichlet (1861) and Riemann (1861). A review of these models has recently been given by Borisov *et al* (2009) who incorporate the later additions concerning gaseous ellipsoids by Ovsiannikov (1956) and by Dyson (1968).

The Dyson model is gravitation-free and represents a rotating and expanding cloud of gas of ellipsoidal shape, with a uniform temperature, a Gaussian density profile and a linear velocity distribution. The vorticity vector is a constant of the motion and, in cases where it vanishes and the gas has the monatomic adiabatic index $\gamma = 5/3$, the Dyson system becomes integrable (see section 2) and is conjectured to have the Painlevé property. This is the particular model that we consider in this paper.

The sub-case where the rotation of the cloud is about a fixed (principal) axis, also referred to as the block-diagonal case, has been treated in detail by Gaffet (2001b, Paper II) who introduced a new set of four variables, denoted by W, ρ, R, π, in terms of which the equations of motion take the remarkable form:

$$M \begin{bmatrix} W' \\ \rho' \\ R' \\ \pi' \end{bmatrix} = 0, \qquad M = \begin{bmatrix} -4(\rho+\pi) & 2W & 0 & \Phi'(W) \\ 2W & -4\alpha^2/3 & -2\pi & R \\ 0 & -2\pi & 1/(4\varepsilon) & \rho \\ \Phi'(W) & R & \rho & -4\beta \end{bmatrix}, \tag{1.1}$$

where a prime denotes the derivative with respect to the (time-like)-independent variable u, and the parameters α^2, ε, β are constants of the motion. Each set of parameters determines a particular (two-dimensional) Liouville torus in phase space, specified by the compatibility condition

$$\det M = 0 \tag{1.2}$$

together with a quadratic constraint relating the four variables:

$$4\beta\pi - \rho R = \Phi(W) \equiv W^2 + 4\varepsilon W + 4\varepsilon. \tag{1.3}$$

Upon substitution of (1.3) into (1.2), it becomes the equation of a *quartic surface* in Cartesian coordinates W, ρ, R, possessing 16 conic point singularities. The data of such a surface are equivalent to those of a sixth degree polynomial P in one variable (Paper II, equations (2.30), (2.31) therein; see also Gaffet 2006, Paper III)—a property at the heart of the *solubility by separation of variables* which was demonstrated in Paper II.

It is of course a question of great interest whether or not the above-mentioned four variables may admit appropriate generalizations when the block-diagonal constraint is relaxed. As a first step toward answering it, let us now consider another sub-case of interest: that of *minimal energy* of the motion, where the energy integral takes its minimum value compatible with a given set of values of the remaining integrals, and the state of rotation is no longer constrained to be about a fixed axis—that is, the 3×3 matrices describing the instantaneous state of motion are not taken to be block-diagonal. For these cases, Gaffet (2006, 2007) obtained a set of four variables playing the role of Cartesian homogeneous coordinates, with respect to which the Liouville torus–(Σ) say—again assumes the form of a quartic surface, possessing 15 conic point singularities (16 in degenerate cases).

These coordinates could only be found, however, through extensive algebraic calculations, and are not related to those of the block-diagonal case in an obvious way. The aim of this paper is to show that they can be replaced by a new set of variables, which is a natural—although highly non-trivial—generalization of the coordinates W, ρ, R. We restrict ourselves to the consideration of the 'degenerate cases', which are those where the polynomial P has a double root, but we hope to extend our results in a future work to the non-degenerate ones as well.

2. The equations of motion and their first integrals

The equations of motion have been given by Dyson (1968), and have been summarized on several occasions by the present author (Gaffet 1996, 2000, 2001c). Basically, the system may be written in the form

$$F_T \ddot{F} = T \tag{2.1}$$

where T is the instantaneous cloud's temperature, F is the 3×3 matrix which determines the linear relation between Lagrangian coordinates **a** and Eulerian coordinates **x**:

$$x^i = F_{ij} a^j, \tag{2.2}$$

the lower index T denotes transposition and the (double) dot denotes (double) differentiation with respect to the time t. Dyson showed that (2.1) may be rewritten in the form

$$\ddot{F}_{ij} + \partial U_{th} / \partial F_{ij} = 0 \tag{2.3}$$

where the thermal energy of the ideal gas U_{th} is proportional to T and, as a consequence of the continuity equation and of the law of adiabatic expansion, is a function of det F only—whence the interpretation as Hamiltonian motion of a particle in the nine-dimensional Euclidean space with coordinates F_{ij}, and the associated integral of energy:

$$E = 1/2 Tr(\dot{F}_T \dot{F}) + U_{th}. \tag{2.4}$$

Gaffet (1996: see section 3.1 therein) has noted that the polar moment of inertia R^2 of *any* freely moving cloud of monatomic gas was a quadratic function of time, and introduced the transformation of the time coordinate:

$$\tau = \int \mathrm{d}t / R^2(t). \tag{2.5}$$

As a result, the motion was shown (Gaffet 1996, sections III.2, III.3) to be still Hamiltonian, with τ playing the role of time, in the particular case of zero angular momentum, which was the main subject of this 1996 article.

Later on, considering in full generality the equations of motion for the Dyson model without vorticity (Gaffet 2001c, equation (4.9) therein), it was found that they coincided with the equations derivable from a Lagrangian (Gaffet 2001a, Paper I, appendix A)—and hence also derivable from the corresponding Hamiltonian, with τ playing the role of time and with the nine coordinates F_{ij} replaced by the new coordinates F_{ij}/R. In this way the radial motion can be eliminated, and the new (Hamiltonian) motion takes place on the eight-dimensional surface of a unit hypersphere.

A detailed treatment of how to derive the new Hamiltonian from the original one may be found in the review by Borisov *et al* (2009).

We now summarize the results concerning the equations of motion and their first integrals.

2.1. Equations of motion

The shape of the ellipsoidal cloud may be represented by a diagonal matrix with unit determinant:

$$\Delta = \mathrm{diag}(\Delta_1, \Delta_2, \Delta_3) \tag{2.6}$$

where Δ_i are proportional to the squares of the principal axes. The instantaneous state of deformation and rotation may be described by a traceless 3×3 velocity matrix v, which is symmetric in the absence of vorticity. Its off-diagonal elements are related to the (anti-symmetric) angular velocity matrix ω in the moving frame—that is, the frame defined by the principal axes:

$$\omega_{ij} = \frac{(\Delta_i + \Delta_j)}{(\Delta_i - \Delta_j)} v_{ij} \qquad (i \neq j) \tag{2.7}$$

while the diagonal elements reflect the rates of deformation as

$$\frac{\mathrm{d}}{\mathrm{d}u} \ln \Delta_i = 2v_{ii}. \tag{2.8}$$

Then the equations may be written compactly in the form

$$\frac{\mathrm{d}v}{\mathrm{d}u} + v^2 + [v, \omega] - \frac{1}{\Delta} = kI \tag{2.9}$$

where k is an *a priori* arbitrary scalar, meaning that the traceless part of the left-hand side must be zero.

As already noted, the equations of motion are derivable from a Hamiltonian, with time τ canonically conjugate to the energy, and it is conjectured—based on the study of various sub-cases—that the system possesses the Painlevé property (Kowalevski 1889a, 1889b, Painlevé 1902, Ince 1956) with respect to the independent variable u:

$$u = \int Tr(\Delta)\, \mathrm{d}\tau. \tag{2.10}$$

2.2. *The constants of motion*

Introducing the notation:

$$\begin{aligned} X_n &= Tr(v^n \Delta) \\ & \qquad\qquad (n = 0, 1, 2) \\ Y_n &= Tr(v^n/\Delta) \end{aligned} \tag{2.11}$$

and the characteristic equation of the matrix v :

$$v^3 + Tv - P = 0, \tag{2.12}$$

the energy constant assumes the form

$$9m = X_0 X_2 - X_1^2 + 3X_0 \tag{2.13}$$

where m is the constant, the term $X_0 X_2 - X_1^2$ represents the kinetic part of the energy, and X_0, which is proportional to the temperature, its potential (thermal) part.

The angular momentum has components in the moving frame:

$$j_k = (\Delta_i - \Delta_j) v_{ij} \qquad (i, j, k = \text{Circ. perm. of } 1, 2, 3) \tag{2.14}$$

and the square j^2 of the angular momentum vector is also constant.

Two more conservation laws, denoted by I_6 and L_6, both of the sixth degree in the momenta, were found in Paper I; the first involves a new vector

$$\tilde{j} = -\Delta j \tag{2.15}$$

which may be viewed as resulting from the vector j under the inversion of the diagonal matrix Δ, and also involves two matricial combinations of v and Δ:

$$U = \Delta^{-1}(v^2 + 4T/3)\Delta^{-1} \qquad V = \Delta(v^2 + T/3) \tag{2.16}$$

with the property

$$\det U = \det V = P^2 + \tfrac{4}{27} T^3. \tag{2.17}$$

It admits the following remarkable form:

$$I_6/27 = \tilde{j}^2/3 + \det U + Tr(2UV + U + 2V). \tag{2.18}$$

The second one assumes the form of a triple product, and involves the new vector $\tilde{j}$ as well as j:

$$L_6 = (\tilde{j}, v\tilde{j}, v^2\tilde{j} - 3j). \tag{2.19}$$

Together with the z-component of angular momentum (in a fixed frame), these constants of the motion ensure Liouville integrability of the system considered (Whittaker 1959).

2.3. Liouville integrability

The system here considered is Liouville integrable (Paper I), that is to say, the five integrals of motion m, J^2, J_z, I_6, L_6 are functionally independent, and the associated symmetry generators commute.

To show that n quantities $E_1, E_2, \ldots, E_n$ are functionally independent, it suffices to show that $E_2 \neq f(E_1), E_3 \neq f(E_1, E_2), \ldots, E_n \neq f(E_1, E_2, \ldots, E_{n-1})$. Clearly, m, J^2, J_z, which represent the energy and angular momentum, should be independent. The fact that functional independence still holds when the new integrals I_6 and L_6 are introduced was proved in Paper I (appendix B). The integral J_z, which is a component of angular momentum in a fixed frame, is evidently functionally independent of all the other integrals, since it involves the rotation matrix relating the fixed and the rotating frames: the rotation matrix, unlike the integrals m, J^2, I_6, L_6, does not admit an expression in terms of quantities defined in the rotating frame.

Concerning the proof that the five integrals commute, we remark that, once the Lagrangian or the Hamiltonian is given (Paper I, appendix A), it is merely a matter of calculation, however long and tedious it may be.

Commutativity of the five integrals has been confirmed independently by Borisov *et al* (2009).

3. The variables π, ρ, R, W and their generalization

Let us first recall that the main motivation for undertaking such a generalization is that these variables are directly related (by a linear transformation with constant coefficients) to variables which make the system manifestly separable in the cases of rotation about a fixed axis.

It is unfortunate that they do not appear to have a clear physical meaning, for otherwise it would probably have greatly simplified the task of deducing their general expression. Nevertheless, it is remarkable that a natural generalization could indeed be found, at least in the cases with precessing motion considered in the present work.

3.1. The block-diagonal definition of the variables

The following definitions were given in Paper I—assuming a block-diagonal matrix v of the general form

$$v = \begin{bmatrix} v_{11} & v_{12} & 0 \\ v_{21} & v_{22} & 0 \\ 0 & 0 & v_{33} \end{bmatrix} \tag{3.1}$$

with $v_{21} = v_{12}$ and $v_{33} = -(v_{11} + v_{22})$:

$$\pi = 1/\Delta_3 \tag{3.2}$$

$$-(\rho + \pi) = v_{33}^2 + T/3 \tag{3.3}$$

$$W = X_2 + TX_0/3 + \pi\rho(\rho + \pi - T) + (\rho + \pi)/\pi \tag{3.4}$$

$$(R/4 + j^2/3)/\varepsilon = -(\Delta_1 + \Delta_2) - (\rho + \pi - T)/\Delta_3 \tag{3.5}$$

and, in the notation of Paper II:

$$\varepsilon = -I_6/108 \qquad \beta = 3\varepsilon m + (1-\varepsilon)j^2/3. \tag{3.6}$$

We remark that the definition of ρ may be rewritten (see equations 2.14, 2.15) as

$$\begin{aligned}\rho &= -(V_{33}+1)/\Delta_3 \\ &= (j|V+1|j)/(j.\tilde{j})\end{aligned} \tag{3.7}$$

while

$$\pi = -j^2/(j.\tilde{j}) \tag{3.8}$$

and the notation $(a|M|b)$, where a, b are vectors and M is a matrix, means the scalar product of a with $M\,b$ (we occasionally also write simply (aMb)).

The above expressions suggest that the four basic variables in the block-diagonal case are really Δ_3, $\rho\Delta_3$, $R\Delta_3$, $W\Delta_3$, which constitute a *Cartesian homogeneous coordinate system* in which the equation of (Σ) has a quartic form.

Thus, we might expect that a generalization of the coordinates should be based on the assumption that $Z_0 = (j.\tilde{j})$ plays the role of a common denominator of the three Cartesian coordinates ρ, R, W, and that the numerator of ρ is $Z_1 = (j|V+1|j)$.

3.2. *Generalizing the variables π and ρ*

Leaving the block-diagonal restriction, we now look for an appropriate generalization of the four variables, under the simplifying assumption of a 'minimal energy' of the motion. We further assume, as a first step, a vanishing value of the integral L_6, which entails degeneracy of the polynomial P associated with the Liouville torus.

In what follows we shall denote by (Σ_4) instead of (Σ) the Liouville torus when expressed in a coordinate system in which it has the form of a quartic surface. Degeneracy entails (see Gaffet 2003) the presence of a double line (L_0) on (Σ)—called the singular solution—so that any plane section of (Σ_4) must be a quartic curve having a double point—a genus 2 curve. As a consequence, under the assumption that $Z_0 = 0$, $Z_1 = 0$ represent plane sections of (Σ_4), they ought to be curves of genus 2 on (Σ). As it turns out, this is not the case, which means that the proposed expressions of Z_0 and Z_1 do not constitute the expected generalizations.

At this point we remark that we may add to Z_n any expression which has the form of a triple product, without altering its value in block-diagonal cases, so that the correct generalized variables might well incorporate such terms. The integral L_6 itself is a triple product but, being merely zero, it does not serve our purpose. It may however be decomposed into a sum of triple products, which are not constant:

$$L_6 = L_{66} + L_{64} \tag{3.9}$$

where the second lower index indicates the (homogeneous) degree in the momenta. Moreover, we may also take into account any other triple product, and in particular those which correspond to the above ones under the exchange of j and $\tilde{j}$, namely

$$\begin{aligned}K_{66} &= (j, vj, v^2 j) \\ K_{64} &= -3(\tilde{j}, j, vj).\end{aligned} \tag{3.10}$$

It then turns out that the following combinations

$$\begin{aligned}Z_0 &= (j.\tilde{j}) + (L_{66} + K_{66})/36 \\ Z_1 &= (j|V+1|j) + (17\,L_{66} + K_{66})/36\end{aligned} \tag{3.11}$$

constitute acceptable candidates for homogeneous Cartesian coordinates, as the sections $Z_0 = 0$, $Z_1 = 0$ are now both of genus 2. The constant factors in the above expressions are for the particular case where

$$m = 5, \qquad j^2 = 12, \qquad \varepsilon = 4,$$

from which it follows that $L_6 = 0$, under the assumption of minimal energy (and in fact, there are only six values of ε compatible with the given values of m and j^2), a case which has been treated in detail by Gaffet (2003). It should be noted that, in view of the fact that the degenerate cases form a homogeneous class, among which the above particular choice is generic, it is clear that any other choice of the constants of motion would at most alter the values of some of the numerical coefficients in the above expressions. We also note that the determination of the genus of the sections considered is facilitated by the existence of a parametric representation of (Σ) (Gaffet 2003, equation (4.4) therein): the sections of genus 2 being characterized by the property that their representation involves a square root of a polynomial of the sixth degree only.

3.3. The variable *R*

Multiplying the right-hand side of the definition (3.5) of R by the common denominator Δ_3 yields the quantity (see equation (2.16))

$$\begin{aligned} Z_{R0}/j^2 &= -(\Delta_1 + \Delta_2)\Delta_3 - (\rho + \pi - T) \\ &= (\Delta_3 - X_0)\Delta_3 + (v_{33}^2 + 4T/3) \\ &= -X_0\Delta_3 + \Delta_3^2(U_{33} + 1) \end{aligned} \tag{3.12}$$

suggesting the following generalized expression:

$$Z_{R0} = (\tilde{j}|U|\tilde{j}) + \tilde{j}^2 + X_0(j.\tilde{j}). \tag{3.13}$$

This is again unsatisfactory, as the section $Z_{R0} = 0$ of the surface (Σ) is found not to be a curve of genus 2. The correct solution in that case turns out to be

$$Z_R = Z_{R0} + (K_{66} - L_{66})/9 \tag{3.14}$$

as the section $Z_R = 0$ is indeed of genus 2 only.

The singular solution is common to the Liouville tori of the block-diagonal case and of the present case incorporating precession: it is their intersection. From the study of the block-diagonal case we know that the linear combination

$$Z_R + Z_1 - 5\, Z_0 \tag{3.15}$$

vanishes on the singular solution, but in the present case it is found to vanish everywhere on the surface (Σ), so that Z_R is merely a linear combination of the two coordinates already obtained.

3.4. The variable *W*

Noting that

$$TrV = X_2 + TX_0/3 \tag{3.16}$$

the block-diagonal definition (3.4) of W may be written as

$$\begin{aligned} W &= Tr\ V - V_{33} - \rho\Delta_3\, U_{33} \\ &= V_{11} + V_{22} - \rho\Delta_3\, U_{33}. \end{aligned} \tag{3.17}$$

Multiplying by $-\Delta_3$ yields (taking account of (3.7)) the quantity

$$Z_{W0}/j^2 = -\Delta_3 Tr\ V + \Delta_3 V_{33} - \Delta_3 U_{33}(V_{33}+1). \tag{3.18}$$

This admits a natural generalization in the form

$$Z_{W0} = (j.\tilde{j})Tr\ V + (\tilde{j}|UV + U - V|j). \tag{3.19}$$

As usual, several triple products of the form (3.10) have to be added in order to obtain a section ($Z_W = 0$) of genus 2:

$$Z_W = Z_{W0} + \tfrac{2}{9}(3\ L_{66} + K_{66} + 2\ K_{64}). \tag{3.20}$$

We choose as our third coordinate the linear combination

$$Z_2 = Z_0 + Z_1 - Z_W/4 \tag{3.21}$$

which is known, from the study of the block-diagonal case, to vanish on the singular solution (L_0).

On (L_0), where the matrix v is block-diagonal, all triple products, such as L_{66}, K_{66}, etc, must vanish. The ratio K_{66}/L_{66} is found to be homographically related to the parameter (denoted by w in Gaffet 2003) occurring in the parametric representation of the surface (Σ):

$$K_{66}/L_{66} = \frac{24}{x} - 5 \tag{3.22}$$

with $x \equiv w + 3$.

The new coordinate Z_2, which must be proportional to L_{66} as well, admits the following simple expression involving x:

$$Z_2 = \tfrac{2}{9}L_{66}\ M_2/x^2 \tag{3.23}$$

where

$$M_2 = x^2 + 24. \tag{3.24}$$

3.5. The fourth coordinate

Owing to the linear dependence of the variable Z_R generalizing R (see equation (3.15)), there is still one coordinate missing.

Let us recall that in Paper III a method has been given to determine a set of four homogeneous coordinates U_n say ($n = 0,1,2,3$) related to the original coordinates by a transformation which involves an *a priori* arbitrary point C on (Σ), called the 'central point' of the transformation, through which all the coordinate surfaces must pass. In order that (Σ) assume a quartic form in these new coordinates, it is necessary however that C should be a singular point on (Σ). Then we find that the (homogeneous) coordinates Z_0, Z_1, Z_2 coincide with the U_n, except for a complicated and non-trivial proportionality factor, *provided that C be taken to be the point at infinity on the singular solution*. (This is in fact the reason for the presence of a factor M_2 in equation (3.23), as this factor vanishes at that point.)

The missing coordinate may then be readily deduced, and is simply given by

$$Z_3 = Z_2/x. \tag{3.25}$$

3.6. The quartic form of the surface (Σ)

The Liouville torus assumes the quartic form, in coordinates Z_0/Z_3, Z_1/Z_3 and $x \equiv Z_2/Z_3$:

$$16N_2(Z_0/Z_3)^2 - 32(x-6)\left(Z_0Z_1/Z_3^2\right) + \tfrac{4}{3}M_2(Z_1/Z_3)^2 - 48x(x-4)(Z_0/Z_3) - 4(x^3-2x^2+96)(Z_1/Z_3) + x^2(x^2+48) = 0 \tag{3.26}$$

with

$$N_2 = x^2 - 6x + 6. \tag{3.27}$$

Moreover, the four coordinates satisfy an identity, the analog of equation (1.3):

$$x^2(3Z_0 + Z_2 - 24) - 2x(12Z_0 + 4Z_1 + 3Z_2) + 48(Z_1 - 12) = 0 \tag{3.28}$$

which is now cubic however, instead of quadratic. It is an inhomogeneous relation, and thus it serves to fix the overall scale of the Z_n.

Let us also note that the above identity provides an independent way of determining the fourth coordinate Z_3 (it is a second degree equation for Z_3).

A reduced form of (Σ_4),

$$N_2\, Z^2 + 12(x-6)Z + 3M_2 - 2D_4X^2 = 0 \tag{3.29}$$

which is quartic in coordinates x, $1/X$, Z/X, may be achieved through a linear transformation

$$\begin{aligned} 1/X &= (x-2) - \tfrac{2}{3}(Z_1/Z_3) \\ Z/X &= 4(Z_0/Z_3 - 3) \end{aligned} \tag{3.30}$$

and

$$D_4 \equiv x^4 - 6x^3 + 18x^2 - 288. \tag{3.31}$$

Equation (3.29) is quartic in coordinates Z, X and $Y \equiv xX$ as well. In that coordinate system the singular solution is the Z axis, and x represents the slope of plane sections through it.

On the singular solution $X = 0$ and $Z \propto 1/\rho$. The resulting second-degree equation for Z has D_4 for discriminant, so that the complete equation (3.29) can be solved for Z in the form

$$N_2\, Z + 6(x-6) = \sqrt{D_4(2N_2X^2 - 3)}. \tag{3.32}$$

4. Common nature of the differential systems in the degenerate cases, with or without precession

In general, the surface (Σ_4) is, in the degenerate cases, a quartic with 16 conic point singularities, 8 of which have coalesced into 4 'double conic points' (see Paper III) located on the singular solution, and its reduced expression must have the form

$$\begin{aligned} &N_2\, Z^2 + 2P_2\, Z + M_2 - k\, D_4\, X^2 = 0 \\ &D_4 \equiv M_2\, N_2 - P_2^2 \end{aligned} \tag{4.1}$$

where M_2, N_2, P_2 are polynomials in x of at most second degree, and k is a reducible constant, which we may take to be unity without loss of generality. Equation (4.1) may be solved for Z as

$$N_2Z + P_2 = \sqrt{D_4(N_2X^2 - 1)}. \tag{4.2}$$

Performing the coordinate transformation on the original equations of motion yields a system where the variable x can be separately determined (see Gaffet 2003, equation (5.3) therein):

$$\mathrm{d}x/\mathrm{d}u = \text{const} \times \sqrt{D_4} \tag{4.3}$$

and the remaining X and Z satisfy a system of the same general form as in the block-diagonal case:

$$\begin{aligned} \mathrm{d}\ln X/\mathrm{d}x &= aZ + b \\ \mathrm{d}Z/\mathrm{d}x &= cX^2 + \mathrm{d}Z + e. \end{aligned} \tag{4.4}$$

In the above equation, a, b, c, d, e are rational functions of x only, which are such that, upon elimination of Z, the resulting differential equation for X assumes the form

$$N_2 \mathrm{d}\ln X/\mathrm{d}x = -N_2'/2 + f(x)\sqrt{(N_2X^2 - 1)/D_4} \tag{4.5}$$

where N_2' is the derivative of the polynomial N_2. Whatever the function f may be, this equation may be rewritten as

$$\mathrm{d}\alpha/\mathrm{d}x = f/N_2/\sqrt{D_4} \tag{4.6a}$$

$$\tan\alpha = \sqrt{N_2X^2 - 1} \tag{4.6b}$$

where α may be geometrically interpreted as the eccentric anomaly on the conic (4.1) (in coordinates $1/X$, Z/X). As the right-hand side of equation (4.6a) is a function of x only, that equation is immediately integrable by quadrature, as expected from a system of the Liouville type.

5. Conclusion

In the paper by Gaffet (2001a, Paper I) it was shown that in the case of rotation about a fixed axis (the block-diagonal case) the evolution of the spinning cloud is governed by a differential system of simple and remarkable form: equation (1.1), in terms of a set of four new variables designated π, ρ, R, W, defined in a quite non-trivial way (section 3.1).

We have shown here that these variables admit a natural generalization when precession is included, at least in the degenerate cases, which are the subject of the present work. The generalization involved three main steps.

- The correct identification of $\Delta_3 \equiv 1/\pi$ as the common denominator of the expressions defining ρ, R, W and, as a consequence, its generalization in the form of the scalar product $j.\tilde{j}$, together with the realization that $(\Delta_3, \rho\Delta_3, R\Delta_3, W\Delta_3)$ play the role of a homogeneous Cartesian coordinate system, in which the Liouville torus assumes a quartic form.
- The (unexpected) occurrence of the triple products L_{66}, K_{66}, K_{64} as additive correction terms to the numerators of ρ, R, W and to their common denominator $j.\tilde{j}$.
- The introduction of two matrices U and V (equation (2.11)), which proved particularly useful when generalizing the variable W (equation (3.19)). Let us mention that they also make possible a remarkably compact (and fully general) formulation of the integral of motion I_6, presented here for the first time (equation (2.18)).

We hope to extend these results in a future work to the non-degenerate cases as well.

Note added. In Paper III we introduced eight auxiliary surfaces given by a polynomial equation $S_n(X_0, Y_0, T) = 0$ (n = 1, …, 8) and we showed that it is possible to choose four linear combinations U_m of them, which constitute a Cartesian coordinate system in which the Liouville torus assumes a quartic form (see also section 3.5 of the present paper). As it turns out, the following seven quantities:

$$1,\ X_0,\ (j\cdot\tilde{j}),\ (jVj),\ \left(Z_{W0} + \tfrac{4}{9}K_{64}\right),\ L_{66},\ K_{66}$$

considered, and defined in full generality here (see equations (3.11), (3.19)), coincide with (linearly independent) linear combinations of the eight ratios S_n/S_1.

This is quite a non-trival result, as these equalities only hold provided that the equation $(F(X_0, Y_0, T) = 0)$ of the Liouville torus is satisfied. The identification of $Z_{W0} + \frac{4}{9}K_{64}$ with one of the polynomials S_n is particularly remarkable, as it involves a rather intricate combination of the matrices U and V (equations (2.16), (3.19)).

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